

# On the stretch factor of the Theta-4 graph

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## Abstract

In this paper we show that the  $\theta$ -graph with 4 cones has constant stretch factor, i.e., there is a path between any pair of vertices in this graph whose length is at most a constant times the Euclidean distance between that pair of vertices. This is the last  $\theta$ -graph for which it was not known whether its stretch factor was bounded.

## 1 Introduction

A  $t$ -spanner of a weighted graph  $G$  is a connected sub-graph  $H$  with the property that for all pairs of vertices  $u$  and  $v$ , the weight of the shortest path between  $u$  and  $v$  in  $H$  is at most  $t$  times the weight of the shortest path between  $u$  and  $v$  in  $G$ , for some fixed constant  $t \geq 1$ . The smallest constant  $t$  for which  $H$  is a  $t$ -spanner of  $G$  is referred to as the *stretch factor* or *spanning ratio* of the graph. The graph  $G$  is referred to as the *underlying graph*. In our setting, the underlying graph is the complete graph on a set of  $n$  points in the plane and the weight of an edge is the Euclidean distance between its endpoints. A spanner of such a graph is called a *geometric spanner*. For a comprehensive overview of geometric spanners, see the book by Narasimhan and Smid [8].

In this paper, we focus on  $\theta$ -graphs. Introduced independently by Clarkson [5] and Keil [7], the  $\theta_m$ -graph is constructed as follows. Given a set  $P$  of points in the plane, we consider each point  $p \in P$  and partition the plane into  $m$  cones (regions in the plane between two rays originating from the same point) with apex  $p$ , each defined by two rays at consecutive multiples of  $\theta = 2\pi/m$  radians from the negative  $y$ -axis. We label the cones  $C_0(p)$  through  $C_{m-1}(p)$ , in counter-clockwise order around  $p$ , starting from the negative  $y$ -axis; see Fig. 1. In each cone  $C_i(p)$ , we add an edge between  $p$  and  $p_i$ , the point in  $C_i(p)$  nearest to  $p$ . However, instead of using the Euclidean distance, we measure distance in  $C_i(p)$  by projecting each vertex onto the angle bisector of this cone. Formally,  $p_i$  is the point in  $C_i(p)$  such that for every other point  $w \in C_i(p)$ , the projection of  $p_i$  onto the angle bisector of  $C_i(p)$  lies closer to  $p$  than that of  $w$ . For simplicity, we assume that no two points of  $P$  lie on a line parallel to either the boundary or the angle bisector of a cone.

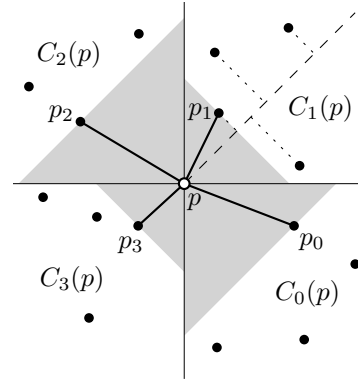


Figure 1: The neighbors of  $p$  in the  $\theta_4$ -graph of  $P$ . Each edge supports an empty isosceles triangle.

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Ruppert and Seidel [9] showed that  $\theta_m$ -graphs are geometric spanners for  $m \geq 7$ , and their stretch factor approaches 1 as  $m$  goes to infinity. Their proof crucially relies on the fact that, given two vertices  $p$  and  $q$  such that  $q \in C_i(p)$ , the distance between  $p_i$  and  $q$  is always less than the distance between  $p$  and  $q$ . This property does not hold for  $m \leq 6$  and indeed, the path obtained by starting at  $p$  and repeatedly following the edge in the cone that contains  $q$ , is not necessarily a spanning path. The main motivation for using spanners is usually to reduce the number of edges in the graph without increasing the length of shortest paths too much. Thus,  $\theta$ -graphs with fewer cones are more interesting in practice, as they have fewer edges. This raises the following question: “What is the smallest  $m$  for which the  $\theta_m$ -graph is a geometric spanner?” Bonichon *et al.* [1] showed that the  $\theta_6$ -graph is also a geometric spanner. Recently, Bose *et al.* [4] proved the same for the  $\theta_5$ -graph. Coming from the other side, El Molla [6] showed that there is no constant  $t$  for which the  $\theta_2$ - and  $\theta_3$ -graphs are geometric spanners. This leaves the  $\theta_4$ -graph as the only open question. Moreover, its resemblance to graphs like the Yao<sub>4</sub>-graph [3] and the  $L_\infty$ -Delaunay triangulation [2], both of which are spanners, make this question more tantalizing. In this paper we establish an upper bound of approximately 237 on the stretch factor of the  $\theta_4$ -graph, thereby showing that it is a geometric spanner. In Section 5, we present a lower bound of 7 that we believe is closer to the true stretch factor of the  $\theta_4$ -graph.

## 2 Existence of a spanning path

Let  $P$  be a set of points in the plane. In this section, we prove that the  $\theta_4$ -graph of  $P$  is a spanner. We do this by showing that the  $\theta_4$ -graph approximates the  $L_\infty$ -Delaunay triangulation. The  $L_\infty$ -Delaunay triangulation of  $P$  is a geometric graph with vertex set  $P$ , and an edge between two points of  $P$  whenever there exists an empty axis-aligned square having these two points on its boundary.

Bonichon *et al.* [2] showed that the  $L_\infty$ -Delaunay triangulation has a stretch factor of  $c^* = \sqrt{4 + 2\sqrt{2}}$ , i.e., there is a path between any two vertices whose length is at most  $c^*$  times their Euclidean distance. We approximate this path in the  $L_\infty$ -Delaunay triangulation by showing the existence of a spanning path in the  $\theta_4$ -graph of  $P$  joining the endpoints of every edge in the  $L_\infty$ -Delaunay triangulation. The main ingredient to obtain this approximation is Lemma 1 whose proof is deferred to Section 4. Before we can state this lemma, we need a few more definitions. Given two points  $s$  and  $t$ , their  $L_1$  distance  $d_{L_1}(s, t)$  is the sum of the absolute differences of their  $x$ - and  $y$ -coordinates.

Let  $S_t(s)$  be the smallest axis-aligned square centered on  $t$  that contains  $s$ . Let  $\ell_t^-$  and  $\ell_t^+$  be the lines with slope  $-1$  and  $+1$  passing through  $t$ , respectively.

Throughout this paper, we repeatedly use  $t$  to denote a *target* point of  $P$  that we want to reach via a path in the  $\theta_4$ -graph. Therefore, we typically omit the reference to  $t$  and write  $\ell^-, \ell^+$  and  $S(s)$  when referring to  $\ell_t^-, \ell_t^+$  and  $S_t(s)$ , respectively.

We say that an object is *empty* if its interior contains no point of  $P$ . An  $s$ - $t$ -path is a path with endpoints  $s$  and  $t$ .

**Lemma 1.** *Let  $s$  and  $t$  be two points of  $P$  such that  $t$  lies in  $C_0(s)$ . If the top-right quadrant of  $S(s)$  is empty and  $C_1(s)$  contains no point of  $P$  below  $\ell^-$ , then there is an  $s$ - $t$ -path in the  $\theta_4$ -graph of  $P$  of length at most  $18 \cdot d_{L_1}(s, t)$ .*

Given a path  $\varphi$ , let  $|\varphi|$  denote the sum of the lengths of the edges in  $\varphi$ . Using Lemma 1, we obtain the following.

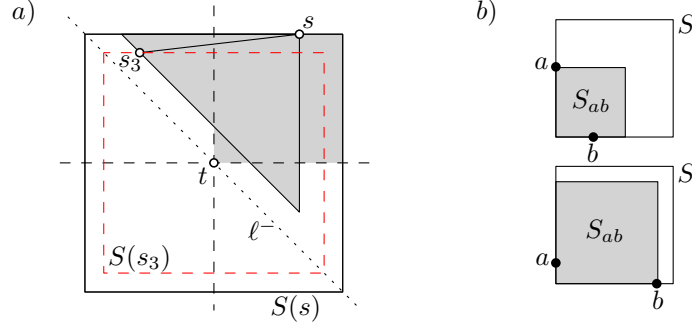


Figure 2: a) Configuration used in the proof of Lemma 2, grey areas represent empty regions. b) If  $a$  and  $b$  lie on consecutive sides of a square  $S$ , there is a square  $S_{ab}$  such that  $ab \subset S_{ab} \subseteq S$  and either  $a$  or  $b$  lies on a corner of  $S_{ab}$ .

**Lemma 2.** *Let  $s$  and  $t$  be two points of  $P$ . If the smallest axis-aligned square enclosing  $s$  and  $t$ , that has  $t$  as a corner, is empty, then there is an  $s$ - $t$ -path in the  $\theta_4$ -graph of  $P$  of length at most  $(\sqrt{2} + 36) \cdot |st|$ .*

*Proof.* Assume without loss of generality that  $s$  lies in  $C_1(t)$ . Then, the top-right quadrant of  $S(s)$  is empty as it coincides with the smallest axis-aligned square enclosing  $s$  and  $t$  that has  $t$  as a corner; see Fig. 2(a). Recall that  $s_3$  is the neighbor of  $s$  in the  $\theta_4$ -graph inside the cone  $C_3(s)$ . Assume that  $s_3 \neq t$  as otherwise the result follows trivially. Consequently,  $s_3$  must lie either in  $C_0(t)$  or in  $C_2(t)$ . Assume without loss of generality that  $s_3$  lies in the top-left quadrant of  $S(s)$ . As  $s_3$  lies in the interior of  $S(s)$ ,  $S(s_3) \subset S(s)$  and hence, the top-right quadrant of  $S(s_3)$  is empty. Moreover,  $s_3$  lies above  $\ell^-$  and hence  $C_1(s_3)$  contains no point of  $P$  below  $\ell^-$ . Therefore, by Lemma 1 there is an  $s_3$ - $t$ -path  $\varphi$  of length at most  $18 \cdot d_{L_1}(s_3, t)$ . Since  $s_3$  lies inside  $S(s)$ ,  $|s_3 t| \leq \sqrt{2} \cdot |st|$  and hence  $|\varphi| \leq 18 \cdot d_{L_1}(s_3, t) \leq 18\sqrt{2} \cdot |s_3 t| \leq 18\sqrt{2}\sqrt{2} \cdot |st| = 36 \cdot |st|$ . Moreover, the length of edge  $ss_3$  is at most  $d_{L_1}(s, t) \leq \sqrt{2} \cdot |st|$  since  $s_3$  must lie above  $\ell^-$ . Thus,  $ss_3 \cup \varphi$  is an  $s$ - $t$ -path of length  $|ss_3| + |\varphi| \leq (\sqrt{2} + 36) \cdot |st|$ .  $\square$

The following observation is depicted in Fig. 2(b).

**Observation 3.** *Let  $S$  be an axis-aligned square. If two points  $a$  and  $b$  lie on consecutive sides along the boundary of  $S$ , then there is a square  $S_{ab}$  containing the segment  $ab$  such that  $S_{ab} \subseteq S$  and either  $a$  or  $b$  lies on a corner of  $S_{ab}$ .*

**Lemma 4.** *Let  $ab$  be an edge of the  $L_\infty$ -Delaunay triangulation of  $P$ . There is an  $a$ - $b$ -path  $\varphi_{ab}$  in the  $\theta_4$ -graph of  $P$  such that  $|\varphi_{ab}| \leq (1 + \sqrt{2}) \cdot (\sqrt{2} + 36) \cdot |ab|$ .*

*Proof.* Let  $T = (a, b, c)$  be a triangle in the  $L_\infty$ -Delaunay triangulation of  $P$ . By definition of this triangulation, there is an empty square  $S$  such that every vertex of  $T$  lies on the boundary of  $S$ . By the general position assumption,  $a, b$  and  $c$  must lie on different sides of  $S$ . If  $a$  and  $b$  lie on consecutive sides of the boundary of  $S$ , then by Observation 3 and Lemma 2 there is a path  $\varphi_{ab}$  contained in the  $\theta_4$ -graph of  $P$  such that  $|\varphi_{ab}| \leq (\sqrt{2} + 36) \cdot |ab|$ .

If  $a$  and  $b$  lie on opposite sides of  $S$ , then both  $ac$  and  $cb$  have their endpoints on consecutive sides along the boundary of  $S$ . Let  $S_{ac}$  be the square contained in  $S$  existing as a consequence of Observation 3 when applied on the edge  $ac$ . Thus, either  $a$  or  $c$  lies on a corner of  $S_{ac}$ . Furthermore, as  $S_{ac}$  is contained in  $S$ , it is also empty. Consequently, by Lemma 2, there is a  $a$ - $c$ -path  $\varphi_{ac}$  such that  $|\varphi_{ac}| \leq (\sqrt{2} + 36) \cdot |ac|$ . Analogously, there is a path  $\varphi_{cb}$  such that  $|\varphi_{cb}| \leq (\sqrt{2} + 36) \cdot |cb|$ . Using elementary geometry, it can be shown

that since  $a$  and  $b$  lie on opposite sides of  $S$ ,  $|ac| + |cb| \leq (1 + \sqrt{2}) \cdot |ab|$ . Therefore, the path  $\varphi_{ab} = \varphi_{ac} \cup \varphi_{cb}$  is an  $a$ - $b$ -path such that  $|\varphi_{ab}| \leq (1 + \sqrt{2}) \cdot (\sqrt{2} + 36) \cdot |ab|$ .  $\square$

**Theorem 5.** *The  $\theta_4$ -graph of  $P$  is a spanner whose stretch factor is at most*

$$(1 + \sqrt{2}) \cdot (\sqrt{2} + 36) \cdot \sqrt{4 + 2\sqrt{2}} \approx 237$$

*Proof.* Let  $\nu$  be the shortest path joining  $s$  with  $t$  in the  $L_\infty$ -Delaunay triangulation of  $P$ . Bonichon *et al.* [2] proved that the length of  $\nu$  is at most  $\sqrt{4 + 2\sqrt{2}} \cdot |st|$ . By replacing every edge in  $\nu$  with the path in the  $\theta_4$ -graph of  $P$  that exists by Lemma 4, we obtain an  $s$ - $t$ -path of length at most

$$(1 + \sqrt{2}) \cdot (\sqrt{2} + 36) \cdot |\nu| \leq (1 + \sqrt{2}) \cdot (\sqrt{2} + 36) \cdot \sqrt{4 + 2\sqrt{2}} \cdot |st| \quad \square$$

### 3 Light paths

We introduce some tools that will help us proving Lemma 1 in Section 4.

Given a point  $p$  of  $P$ , we call edge  $pp_i$  an  $i$ -edge. Let  $\varphi$  be a path that follows only 0- and 1-edges. A 0-edge  $pp_0$  of  $\varphi$  is *light* if no edge of  $\varphi$  crosses the horizontal ray shooting to the right from  $p$ . We say that  $\varphi$  is a *light* path if all its 0-edges are light. In this section we show how to bound the length of a light path with respect to the Euclidean distance between its endpoints.

Notice that every  $i$ -edge is associated with an empty isosceles right triangle. For a point  $p$ , the empty triangle generated by its  $i$ -edge is denoted by  $\Delta_i(p)$ .

**Lemma 6.** *Given a light path  $\varphi$ , every pair of 0-edges of  $\varphi$  has disjoint orthogonal projection on the line defined by the equation  $y = -x$ .*

*Proof.* Let  $s$  and  $t$  be the endpoints of  $\varphi$ . Let  $pp_0$  be any 0-edge of  $\varphi$  and let  $\nu_{p_0}$  be the diagonal line extending the hypotenuse of  $\Delta_0(p)$ , i.e.,  $\nu_{p_0}$  is a line with slope  $+1$  passing through  $p_0$ . Let  $\gamma$  be the path contained in  $\varphi$  that joins  $p_0$  with  $t$ . We claim that every point in  $\gamma$  lies below  $\nu_{p_0}$ . If this claim is true, the diagonal lines constructed from the empty triangles of every 0-edge in  $\varphi$  split the plane into disjoint slabs, each containing a different 0-edge of  $\varphi$ . Thus, their projection on the line defined by the equation  $y = -x$  must be disjoint.

To prove that every point in  $\gamma$  lies below  $\nu_{p_0}$ , notice that every point in  $\gamma$  must lie to the right of  $p$  since  $\varphi$  is  $x$ -monotone, and below  $p$  since  $pp_0$  is light, i.e.,  $\gamma$  is contained in  $C_0(p)$ . Since  $\Delta_0(p)$  is empty, no point of  $\gamma$  lies above  $\nu_{p_0}$  and inside  $C_0(p)$  yielding our claim.  $\square$

Given a point  $w$  of  $P$ , we say that a point  $p$  of  $P$  is  $w$ -protected if  $C_1(p)$  contains no point of  $P$  below or on  $\ell_w^-$ , recall that  $\ell_w^-$  is the line with slope  $-1$  passing through  $w$ . In other words, a point  $p$  is  $w$ -protected if either  $C_1(p)$  is empty or  $p_1$  lies above  $\ell_w^-$ . Moreover, every point lying above  $\ell_w^-$  is  $w$ -protected and no point in  $C_3(w)$  is  $w$ -protected.

Given two point  $s$  and  $t$  such that  $s$  lies to the left of  $t$ , we aim to construct a path joining  $s$  with  $t$  in the  $\theta_4$ -graph of  $P$ . The role of  $t$ -protected points will be central in this construction. However, as a first step, we relax our goal and prove instead the existence of a light path  $\sigma_{s \rightarrow t}$  going from  $s$  towards  $t$  that does not necessarily end at  $t$ .

To construct  $\sigma_{s \rightarrow t}$ , start at a point  $z = s$  and repeat the following steps until reaching either  $t$  or a  $t$ -protected point  $w$  lying to the right of  $t$ .

- If  $z$  is not  $t$ -protected, then follow its 1-edge, i.e., let  $z = z_1$ .

- If  $z$  is  $t$ -protected, then follow its 0-edge, i.e., let  $z = z_0$ .

The pseudocode of this algorithm can be found in Algorithm 1.

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**Algorithm 1** Given two points  $s$  and  $t$  of  $P$  such that  $s$  lies to the left of  $t$ , algorithm to compute the path  $\sigma_{s \rightarrow t}$

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1: Let  $z = s$ .
2: Append  $s$  to  $\sigma_{s \rightarrow t}$ .
3: while  $z \neq t$  and  $z$  is not a  $t$ -protected point lying to the right of  $t$  do
4:   if  $z$  is  $t$ -protected then  $z = z_0$  else  $z = z_1$ 
5:   Append  $z$  to  $\sigma_{s \rightarrow t}$ .
6: end while
7: return  $\sigma_{s \rightarrow t}$ 

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**Lemma 7.** *Let  $s$  and  $t$  be two points of  $P$  such that  $s$  lies to the left of  $t$ . Algorithm 1 produces a light  $x$ -monotone path  $\sigma_{s \rightarrow t}$  joining  $s$  with a  $t$ -protected point  $w$  such that either  $w = t$  or  $w$  lies to the right of  $t$ . Moreover, every edge on  $\sigma_{s \rightarrow t}$  is contained in  $S(s)$ .*

*Proof.* By construction, Algorithm 1 finishes only when reaching either  $t$  or a  $t$ -protected point lying to the right of  $t$ . Since every edge of  $\sigma_{s \rightarrow t}$  is either a 0-edge or a 1-edge traversed from left to right,  $\sigma_{s \rightarrow t}$  is  $x$ -monotone.

The left endpoint of every 0-edge in  $\sigma_{s \rightarrow t}$  lies in  $C_2(t)$  as it must be  $t$ -protected and no  $t$ -protected point lies in  $C_3(t)$ . Thus, if  $vv_0$  is a 0-edge, then  $v$  lies in  $C_2(t)$  and hence,  $v_0$  lies inside  $S(s)$  and above  $\ell^+$ . Otherwise  $t$  would lie inside  $\Delta_0(v)$ . Therefore, every 0-edge in  $\sigma_{s \rightarrow t}$  is contained in  $S(s)$ .

Every 1-edge in  $\sigma_{s \rightarrow t}$  has its two endpoints lying below  $\ell^-$ ; otherwise, we followed the 1-edge of a  $t$ -protected point which is not allowed by Step 4 of Algorithm 1. Thus, every 1-edge in  $\sigma_{s \rightarrow t}$  lies below  $\ell^-$  and to the right of  $s$ . As 1-edges are traversed from bottom to top and the 0-edges of  $\sigma_{s \rightarrow t}$  are enclosed by  $S(s)$ , every 1-edge in  $\sigma_{s \rightarrow t}$  is contained in  $S(s)$ .

Let  $vv_0$  be any 0-edge of  $\sigma_{s \rightarrow t}$ . Since we followed the 0-edge of  $v$ , we know that  $v$  is  $t$ -protected and hence no point of  $P$  lies in  $C_1(v)$  and below  $\ell^-$ . As every 1-edge has its two endpoints lying below  $\ell^-$  and  $\sigma_{s \rightarrow t}$  is  $x$ -monotone, no 1-edge in  $\sigma_{s \rightarrow t}$  can have an endpoint in  $C_1(v)$ . In addition, every 0-edge of  $\sigma_{s \rightarrow t}$  joins its left endpoint with a point below it. Thus, no 0-edge of  $\sigma_{s \rightarrow t}$  can cross the ray shooting to the right from  $v$ . Consequently,  $vv_0$  is light and hence  $\sigma_{s \rightarrow t}$  is a light path; see Fig 3.  $\square$

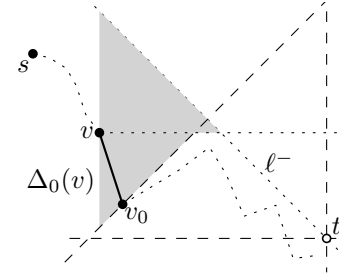


Figure 3: If  $v$  is a  $t$ -protected point, then edge  $vv_0$  is light in any path  $\sigma_{s \rightarrow t}$  that contains it.

Given two points  $p$  and  $q$ , let  $|pq|_x$  and  $|pq|_y$  be the absolute differences between their  $x$ - and  $y$ -coordinates, respectively, i.e.,  $d_{L_1}(p, q) = |pq|_x + |pq|_y$ .

**Lemma 8.** *Let  $s$  and  $t$  be two points of  $P$  such that  $s$  lies to the left of  $t$ . If  $s$  is  $t$ -protected, then  $|\sigma_{s \rightarrow t}| \leq 3 \cdot d_{L_1}(s, t)$ .*

*Proof.* To bound the length of  $\sigma_{s \rightarrow t}$ , we bound the length of its 0-edges and the length of its 1-edges separately. Let  $Z$  be the set of all 0-edges in  $\sigma_{s \rightarrow t}$  and consider their orthogonal projection on  $\ell^-$ . By Lemma 6, all these projections are disjoint. Moreover, the length of every 0-edge in  $Z$  is at most  $\sqrt{2}$  times the length of its projection. Let  $s_\perp$  be the

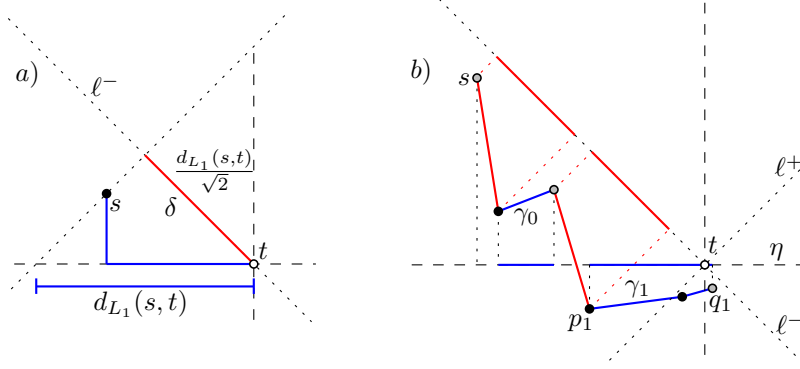


Figure 4: a) The segment  $\delta$  having length  $d_{L_1}(s, t)/\sqrt{2}$ . b) The 0-edges of  $\sigma_{s \rightarrow t}$  have disjoint projections on  $\ell^-$  and the 1-edges have disjoint projections on the horizontal line passing through  $t$ . The slope between the endpoints of the maximal paths  $\gamma_0$  and  $\gamma_1$  is less than 1.

orthogonal projection of  $s$  on  $\ell^-$  and let  $\delta$  be the segment joining  $s_\perp$  with  $t$ . Since  $s$  is  $t$ -protected and  $\sigma_{s \rightarrow t}$  is  $x$ -monotone, the orthogonal projection of every 0-edge of  $Z$  on  $\ell^-$  is contained in  $\delta$  and hence  $\sum_{e \in Z} |e| \leq \sqrt{2} \cdot |\delta|$ . Since  $|\delta| = d_{L_1}(s, t)/\sqrt{2}$  as depicted in Fig. 4(a), we conclude that  $\sum_{e \in Z} |e| \leq d_{L_1}(s, t)$ .

Let  $O$  be the set of all 1-edges in  $\sigma_{s \rightarrow t}$  and let  $\eta$  be the horizontal line passing through  $t$ . Since  $\sigma_{s \rightarrow t}$  is  $x$ -monotone, the orthogonal projections of all edges in  $O$  on  $\eta$  are disjoint. Let  $\gamma_0, \dots, \gamma_k$  be the connected components induced by  $O$ , i.e., the set of maximal connected paths that can be formed by the 1-edges in  $O$ ; see Fig. 4(b). We claim that the slope of the line joining the two endpoints  $p^i, q^i$  of every  $\gamma_i$  is smaller than 1. If this claim is true, the length of every  $\gamma_i$  is bounded by  $|p^i q^i|_x + |p^i q^i|_y \leq 2 \cdot |p^i q^i|_x$  as each  $\gamma_i$  is  $x$ - and  $y$ -monotone.

To prove that the slope between  $p^i$  and  $q^i$  is smaller than 1, let  $vv_0$  be the 0-edge of  $\sigma_{s \rightarrow t}$  such that  $v_0 = p^i$ . Since  $vv_0$  is in  $\sigma_{s \rightarrow t}$ ,  $v$  is  $t$ -protected by Step 4 of Algorithm 1 and hence, as  $\Delta_0(v)$  is empty,  $q^i$  must lie below the line with slope +1 passing through  $p^i$  yielding our claim.

Let  $\omega$  be the segment obtained by shooting a ray from  $t$  to the left until hitting the boundary of  $S(s)$ . We bound the length of all edges in  $O$  using the length of  $\omega$ . Notice that the orthogonal projection of every  $\gamma_i$  on  $\eta$  is contained in  $\omega$ , except maybe for  $\gamma_k$  whose right endpoint  $q^k$  could lie below and to the right of  $t$ . Two cases arise: If the projection of  $\gamma_k$  on  $\eta$  is contained in  $\omega$ , then  $\sum_{i=0}^k |\gamma_i| \leq \sum_{i=0}^k 2 \cdot |p^i q^i|_x \leq 2 \cdot |\omega|$ . Otherwise, since  $q_k$  is  $t$ -protected,  $q_k$  lies below  $\ell^-$  and hence  $d_{L_1}(p^k, q^k) \leq d_{L_1}(p^k, t)$ . Moreover,  $p^k$  must lie above  $\ell^+$  as  $p^k$  is reached by a 0-edge coming from above  $\eta$ , i.e.,  $|p^k t|_y < |p^k t|_x$ . Therefore,

$$|\gamma_k| \leq d_{L_1}(p^k, q^k) \leq d_{L_1}(p^k, t) = |p^k t|_x + |p^k t|_y \leq 2 \cdot |p^k t|_x$$

Consequently,  $\sum_{i=0}^k |\gamma_i| \leq 2 \cdot |p^k t|_x + \sum_{i=0}^{k-1} 2 \cdot |p^i q^i|_x \leq 2 \cdot |\omega|$ . Since  $|\omega| \leq d_{L_1}(s, t)$ , we get that  $\sum_{e \in O} |e| = \sum_{i=0}^k |\gamma_i| \leq 2 \cdot d_{L_1}(s, t)$ . Thus,  $\sigma_{s \rightarrow t}$  is a light path of length at most  $\sum_{e \in O} |e| + \sum_{e \in Z} |e| \leq 3 \cdot d_{L_1}(s, t)$ .  $\square$

By the construction of the light path in Algorithm 1, we observe the following.

**Lemma 9.** *Let  $s$  and  $t$  be two points of  $P$  such that  $s$  lies to the left of  $t$ . If the right endpoint  $w$  of  $\sigma_{s \rightarrow t}$  is not equal to  $t$ , then  $w$  lies either above  $\ell^+$  if  $w \in C_1(t)$ , or below  $\ell^-$  if  $w \in C_0(t)$ .*



**Case 1.** Assume that  $s_0$  lies inside  $R$ . If  $s_0$  lies above  $\ell^-$ , then  $s_0$  is  $t$ -protected and hence we are done after applying our induction hypothesis on  $s_0$ . If  $s_0$  lies below  $\ell^-$ , then we can follow its  $\max_1$ -path to reach a  $t$ -protected point  $w$  that must lie inside  $R$  as  $s$  is  $t$ -protected. By running Algorithm 1 on  $s$  and  $w$ , we obtain a path  $\sigma_{s \rightarrow w}$  that goes through the edge  $ss_0$  and then follows the  $\max_1$ -path of  $s_0$  until reaching  $w$ ; see Fig. 6.

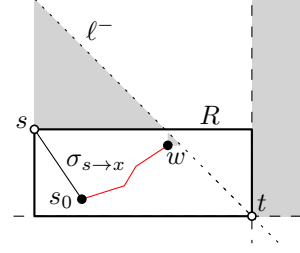


Figure 6: Case 1.

Since  $s$  is  $t$ -protected and  $w$  lies below  $\ell^-$ ,  $s$  is also  $w$ -protected. Therefore, Lemma 8 guarantees that  $|\sigma_{s \rightarrow w}| \leq 3 \cdot d_{L_1}(s, w)$ . By induction hypothesis on  $w$ , there is a  $w$ - $t$ -path  $\varphi$  such that  $|\varphi| \leq 18 \cdot d_{L_1}(w, t)$ . As  $w$  lies in  $R$ , by joining  $\sigma_{s \rightarrow w}$  with  $\varphi$  we obtain the desired  $s$ - $t$ -path of length at most  $18 \cdot d_{L_1}(s, t)$ .

**Case 2.** Assume that  $s_0$  does not lie in  $R$ . This implies that  $s_0$  lies below  $t$ . Assume also that  $\sigma_{s \rightarrow t}$  does not reach  $t$ ; otherwise we are done since  $|\sigma_{s \rightarrow t}| \leq 3 \cdot d_{L_1}(s, t)$ . Thus, as the top-right quadrant of  $S(s)$  is empty,  $\sigma_{s \rightarrow t}$  ends at a  $t$ -protected point  $z$  lying in the bottom-right quadrant of  $S(s)$ . We consider two sub-cases depending on whether  $\sigma_{s \rightarrow t}$  contains a point inside  $R$  or not.

**Case 2.1.** If  $\sigma_{s \rightarrow t}$  contains a point inside  $R$ , let  $w$  be the first  $t$ -protected point of  $\sigma_{s \rightarrow t}$  after  $s$  and note that  $w$  also lies inside  $R$  since  $s$  is  $t$ -protected. Notice that the part of  $\sigma_{s \rightarrow t}$  going from  $s$  to  $w$  is in fact equal to  $\sigma_{s \rightarrow w}$  since  $w$  lies above  $t$  and only 1-edges were followed after  $s_0$  by Step 4 of Algorithm 1; see Fig. 7. Thus, as  $s$  is also  $w$ -protected, the length of  $\sigma_{s \rightarrow w}$  is bounded by  $3 \cdot d_{L_1}(s, w)$  by Lemma 8. Hence, we can apply the induction hypothesis on  $w$  as before and obtain the desired  $s$ - $t$ -path.

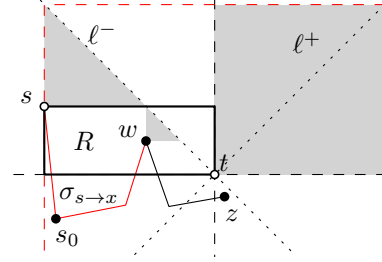


Figure 7: Case 2.1.

**Case 2.2.** If  $\sigma_{s \rightarrow t}$  does not contain a point inside  $R$ , then  $\sigma_{s \rightarrow t}$  follows only 1-edges from  $s_0$  until reaching  $z$  in the bottom-right quadrant of  $S(s)$ ; see Fig. 8(a). Let  $P^*$  be the set of points obtained by reflecting  $P$  on the line  $\ell^+$ . Since  $z$  remains  $t$ -protected after the reflection, we can use Algorithm 1 to produce a path  $\sigma_{z \rightarrow t}^*$  in the  $\theta_4$ -graph of  $P^*$ . Let  $\gamma_{z \rightarrow t}$  be the path in the  $\theta_4$  graph of  $P$  obtain by reflecting  $\sigma_{z \rightarrow t}^*$  on  $\ell^+$ . Note that  $\gamma_{z \rightarrow t}$  ends at a point  $w$  such that  $w$  is either equal to  $t$  or  $w$  lies in the top-left quadrant of  $S(s)$  since the top-right quadrant of  $S(s)$  is empty. Since  $z$  lies inside  $S(s)$ ,  $d_{L_1}(z, t) \leq 2 \cdot d_{L_1}(s, t)$ . Therefore, by Lemma 8, the length of  $\sigma_{s \rightarrow t} \cup \gamma_{z \rightarrow t}$  is given by

$$|\sigma_{s \rightarrow t}| + |\gamma_{z \rightarrow t}| \leq 3 \cdot d_{L_1}(s, t) + 3 \cdot d_{L_1}(z, t) \leq 3 \cdot d_{L_1}(s, t) + 6 \cdot d_{L_1}(s, t) = 9 \cdot d_{L_1}(s, t).$$

Two cases arise: If  $\gamma_{z \rightarrow t}$  reaches  $t$  ( $w = t$ ), then we are done since  $\sigma_{s \rightarrow t} \cup \gamma_{z \rightarrow t}$  joins  $s$  with  $t$  through  $z$ .

If  $\gamma_{z \rightarrow t}$  does not reach  $t$  ( $w \neq t$ ), then  $w$  lies below  $\ell^-$  by Lemma 9 applied on  $\sigma_{z \rightarrow t}^*$ . Moreover, as  $s$  is  $t$ -protected, no point in  $C_1(s)$  can be reached by  $\gamma_{z \rightarrow t}$  and hence  $w$  must lie inside  $R$ . We claim that  $d_{L_1}(s, t) \leq 2 \cdot d_{L_1}(s, w)$ . If this claim is true,  $|\sigma_{s \rightarrow t} \cup \gamma_{z \rightarrow t}| \leq 9 \cdot d_{L_1}(s, t) \leq 18 \cdot d_{L_1}(s, w)$ . Furthermore, by the induction hypothesis, there is a path  $\varphi$  joining  $w$  with  $t$  of length at most  $18 \cdot d_{L_1}(w, t)$ . Consequently, by joining  $\sigma_{s \rightarrow t}$ ,  $\gamma_{z \rightarrow t}$  and  $\varphi$ , we obtain an  $s$ - $t$ -path of length at most  $18 \cdot d_{L_1}(s, w) + 18 \cdot d_{L_1}(w, t) = 18 \cdot d_{L_1}(s, t)$ .



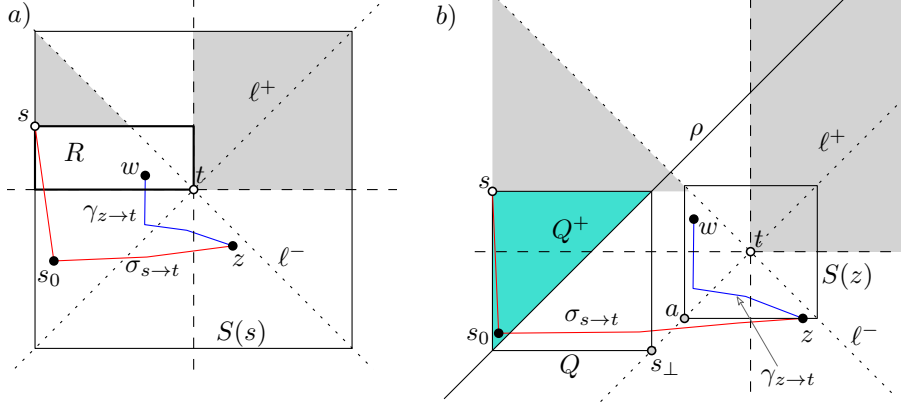


Figure 8: a) Case 2.2 in the proof of Lemma 1, path  $\sigma_{s \rightarrow t}$  has no point inside  $R$  and reaches a point  $z$  lying in the bottom-right quadrant of  $S(s)$ . b) The inductive argument proving that the point  $w$ , reached after taking the path  $\gamma_{z \rightarrow t}$ , lies outside of the triangle  $Q^+$  containing all the points above  $\rho$  and below  $s$ . As  $s$  is  $t$ -protected, the region above  $s$  and below  $\rho$  is empty.

To prove that  $d_{L_1}(s, t) \leq 2 \cdot d_{L_1}(s, w)$ , let  $s_\perp$  be the orthogonal projection of  $s$  on  $\ell^+$ . Let  $\rho$  be the perpendicular bisector of the segment  $ss_\perp$  and notice that for every point  $y$  in  $C_0(s)$ ,  $d_{L_1}(s, t) \leq 2 \cdot d_{L_1}(s, y)$  if and only if  $y$  lies below  $\rho$ .

Let  $Q$  be the minimum axis-aligned square containing  $s$  and  $s_\perp$ . Note that  $\rho$  splits  $Q$  into two equal triangles  $Q^+$  and  $Q^-$  as one diagonal of  $Q$  is contained in  $\rho$ . Assume that  $Q^+$  is the triangle that lies above  $\rho$ . Notice that all points lying in  $C_0(s)$  and above  $\rho$  are contained in  $Q^+$ ; see Fig. 8(b). We prove that  $w$  lies outside of  $Q^+$  and hence, that  $w$  must lie below  $\rho$ .

If  $s_0$  lies below  $\rho$ , then the empty triangle  $\Delta_0(s)$  contains  $Q^+$  forcing  $w$  to lie below  $\rho$ . Assume that  $s_0$  lies above  $\rho$ . In this case,  $z$  lies above  $s_0$  as we only followed 1-edges to reach  $z$  in the construction of  $\sigma_{s \rightarrow t}$  by Step 4 of Algorithm 1. Let  $a$  be the intersection of  $\ell^+$  and the ray shooting to the left from  $z$ . Notice that  $w$  must lie to the right of  $a$  as the path  $\gamma_{z \rightarrow t}$  is contained in the square  $S(z)$  and  $a$  is one of its corners. As  $z$  lies above  $s_0$  and  $s_0$  lies above  $s_\perp$ , we conclude that  $a$  is above  $s_\perp$  and both lie on  $\ell^+$ . Therefore,  $a$  lies to the right of  $s_\perp$ , implying that  $w$  lies to the right of  $s_\perp$  and hence outside of  $Q^+$ . As we proved that  $w$  lies below  $\rho$ , we conclude that  $d_{L_1}(s, t) \leq 2 \cdot d_{L_1}(s, w)$ .  $\square$

## 5 Lower Bound

In this section we show how to construct a lower bound of 7 for the  $\theta_4$ -graph. We start with two vertices  $u$  and  $w$  such that  $w$  lies in  $C_2(u)$  and the difference of their  $x$ -coordinates is arbitrarily small. To construct the lower bound, we repeatedly replace a single edge of the shortest  $u$ - $w$ -path by placing vertices in the corners of the empty triangle(s) associated with that edge. The final graph is shown in Fig. 9.

We start out by removing the edge between  $u$  and  $w$  by placing two vertices, one inside  $\Delta_2(u)$  and one inside  $\Delta_0(w)$ , both arbitrarily close to the corner that does not contain  $u$  nor  $w$ . Let  $v_1$  be the vertex placed in  $\Delta_2(u)$ . Placing  $v_1$  and the other vertex in  $\Delta_0(w)$  removed edge  $uw$ , but created two new shortest paths,  $uv_1w$  being one of them. Hence, our next step is to extend this path.

We remove edge  $v_1w$  (and its equivalent in the other path) by placing a vertex arbitrarily close to the corner of  $\Delta_1(v_1)$  and  $\Delta_3(w)$  that is farthest from  $u$ . Let  $v_2$  be the vertex placed inside  $\Delta_1(v_1)$ . Hence, edge  $v_1w$  is replaced by the path  $v_1v_2w$ .

Next, we extend the path again by removing edge  $v_2w$  (and its equivalent edge in the other paths). Like before, we place a vertex arbitrarily close to the corner of  $\Delta_0(v_2)$  and  $\Delta_2(w)$  that is farthest from  $u$ . Let  $v_3$  be the vertex placed in  $\Delta_0(v_2)$ . Hence, edge  $v_2w$  is replaced by  $v_2v_3w$ .

Finally, we replace edge  $v_3w$  (and its equivalent edge in the other paths). For all paths for which this edge lies on the outer face, we place a vertex in the corner of the two empty triangles defining that edge. However, for edge  $v_3w$  which does not lie on the outer face, we place a single vertex  $v_4$  in the intersection of  $\Delta_3(v_3)$  and  $\Delta_1(w)$ . In this way, edge  $v_3w$  is replaced by  $v_3v_4w$ . When placing  $v_4$ , we need to ensure that no edge  $uv_4$  is added as this would create a shortcut. This is easily achieved by placing  $v_4$  such that it is closer to  $v_3$  than to  $w$ . The resulting graph is shown in Fig. 9.

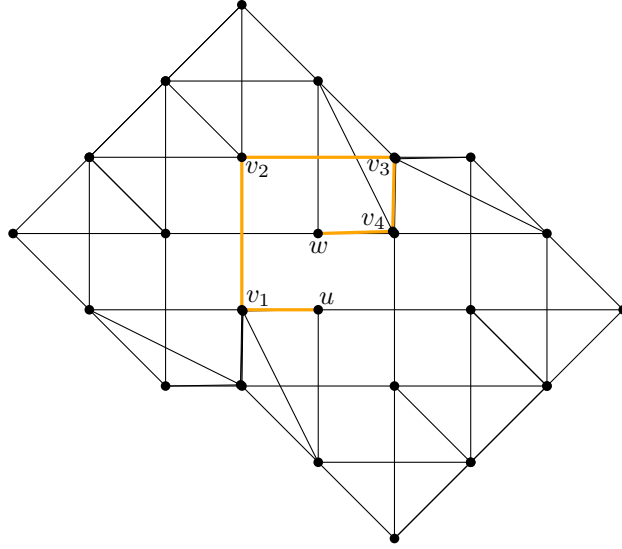


Figure 9: A lower bound for the  $\theta_4$ -graph. One of the shortest paths from  $u$  to  $w$  goes via  $v_1$ ,  $v_2$ ,  $v_3$ , and  $v_4$ .

**Lemma 10.** *The stretch factor of the  $\theta_4$ -graph is at least 7.*

*Proof.* We look at path  $uv_1v_2v_3v_4w$  from Fig. 9. Edges  $uv_1$ ,  $v_3v_4$ , and  $v_4w$  have length  $|uw| - \varepsilon$  and edges  $v_1v_2$  and  $v_2v_3$  have length  $2 \cdot |uw| - \varepsilon$ , where  $\varepsilon$  is positive and arbitrarily close to 0. Hence the stretch factor of this path is arbitrarily close to 7.  $\square$

## References

- [1] N. Bonichon, C. Gavoille, N. Hanusse, and D. Ilcinkas. Connections between theta-graphs, Delaunay triangulations, and orthogonal surfaces. In *Proceedings of the 36th International Conference on Graph Theoretic Concepts in Computer Science (WG 2010)*, pages 266–278, 2010.
- [2] N. Bonichon, C. Gavoille, N. Hanusse, and L. Perković. The stretch factor of  $L_1$ - and  $L_\infty$ -Delaunay triangulations. In *Proceedings of the European Symposia on Algorithms (ESA 2012)*, pages 205–216, 2012.
- [3] P. Bose, M. Damian, K. Douïeb, J. O’Rourke, B. Seamone, M. Smid, and S. Wührer.  $\pi/2$ -angle Yao graphs are spanners. *International Journal of Computational Geometry & Applications*, 22(1):61–82, 2012.

- [4] P. Bose, P. Morin, A. van Renssen, and S. Verdonschot. The  $\theta_5$ -graph is a spanner. *Computer Research Repository (CoRR)*, abs/1212.0570, 2012.
- [5] K. Clarkson. Approximation algorithms for shortest path motion planning. In *Proceedings of the 19th Annual ACM Symposium on Theory of Computing (STOC 1987)*, pages 56–65, 1987.
- [6] N. M. El Molla. Yao spanners for wireless ad hoc networks. Master’s thesis, Villanova University, 2009.
- [7] J. Keil. Approximating the complete Euclidean graph. In *Proceedings of the 1st Scandinavian Workshop on Algorithm Theory (SWAT 1988)*, pages 208–213, 1988.
- [8] G. Narasimhan and M. Smid. *Geometric Spanner Networks*. Cambridge University Press, 2007.
- [9] J. Ruppert and R. Seidel. Approximating the  $d$ -dimensional complete Euclidean graph. In *Proceedings of the 3rd Canadian Conference on Computational Geometry (CCCG 1991)*, pages 207–210, 1991.